NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS

2015 HSC Course Assessment Task 4 (Trial Examination)
Thursday July 30, 2015

General instructions

- Working time – 3 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 13)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

BOSTES NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M6 – Mrs Bhamra

◯ 12M3 – Mr Wall

◯ 12M4 – Mr Tan

◯ 12M5 – Mr Lam

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	Total
MARKS	/10	/15	/15	/15	/15	/15	/15	/100

Marking grid

Question	YO1	YO2	YO3	YO4	YO5	Total
MCQ	6 —/1	2, 4, 7, 10 —/4	8, 9 —/2	1, 3 —/2	5 —/1	—/10
11	a, c, d, e —/9	b —/1		f, g —/5		—/15
12	a —/2	b, c(i), d —/9	c(ii)(iii) —/4			—/15
13		a, b —/8		c, d —/7		—/15
14			a —/6		b —/9	—/15
15			a —/3	b, c —/12		—/15
16	a(ii)(iii), b —/10	a(i) —/1	c —/4			—/15
Total	—/22	—/23	—/19	—/26	—/10	—/100

Section I

10 marks
Attempt Question 1 to 10
Mark your answers on the answer grid provided (labelled as page 13).

Questions **Marks**

1. Find the value of $\lim_{x \to -3} \frac{x^3 + 27}{x^2 - x - 12}$. **1**

(A) -27 (C) -9

(B) $-\frac{27}{7}$ (D) $-\frac{9}{7}$

2. Which of the following represents the range of $y = \frac{1}{2} - \frac{1}{2}\cos 2x$? **1**

(A) $-\frac{1}{2} \le y \le 0$ (B) $-\frac{1}{2} \le y \le \frac{1}{2}$ (C) $0 \le y \le \frac{1}{2}$ (D) $0 \le y \le 1$

3. What is the primitive of $x^{-2} + 6x$? **1**

(A) $-\frac{1}{x} + 3x^2$ (B) $\frac{1}{x} + 3x^2$ (C) $-\frac{2}{x^3} + 6$ (D) $\frac{2}{x^3} + 6$

4. How many solutions are there to $\cos 2x = \frac{\sqrt{3}}{2}$ within the interval $0 \le x \le 2\pi$? **1**

(A) 1 (B) 2 (C) 3 (D) 4

5. For the curve $y = f(x)$, it is known that $f''(x) = (x+4)^2 (x-2)$. **1**

Which of the following statements is true?

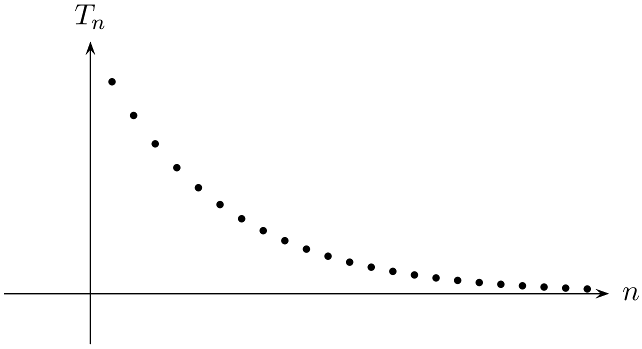
(A) $x = -4$ and $x = 2$ are both the x coordinates of a point of inflexion.

(B) $x = -4$ is the only x coordinate of a point of inflexion.

(C) $x = 2$ is the only x coordinate of a point of inflexion.

(D) Neither $x = -4$ or $x = 2$ are the x coordinates of a point of inflexion.

6. The graph shows consecutive terms of a sequence. Which of the following statements best describes this sequence? **1**



(A) geometric, $|r| < 1$

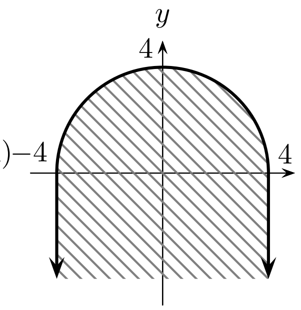
(B) geometric, $|r| \geq 1$

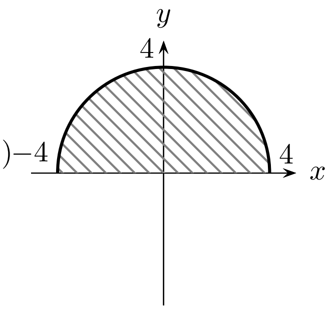
(C) arithmetic, $|d| < 1$

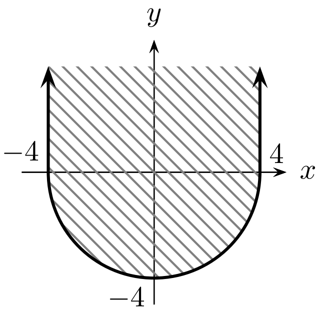
(D) arithmetic, $|d| \geq 1$

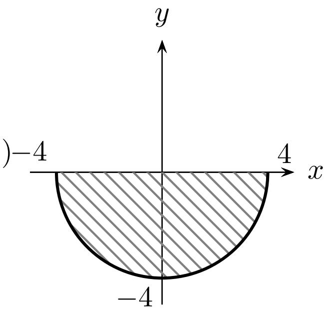
7. Which of the following represents the inequality **1**

$$y \leq \sqrt{4 - x^2}$$

(A)


(C)


(B)


(D)


8. Which of the following is numerically equivalent to $\log_3 15$? **1**

(A) $\dfrac{\log_{15} 3}{\log_{15} 5}$ (B) $\dfrac{\log_5 3}{\log_5 15}$ (C) $\dfrac{\log_e 3}{\log_e 15}$ (D) $\dfrac{\log_5 15}{\log_5 3}$

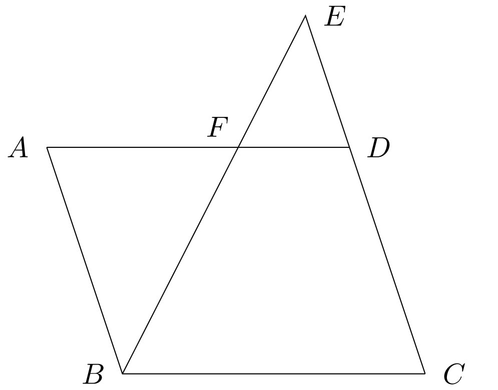
9. What is the value of q if **1**

$$x^2 + p(x+5) + q \equiv (x-2)(x+5)$$

(A) -25 (B) -10 (C) 3 (D) 5

10. In the following diagram, $ABCD$ is a parallelogram. F is a point lying on AD. BF produced and CD produced meet at E. **1**

If $CD : DE = 2 : 1$, then what is the ratio of $AF : BC$?



(NOT TO SCALE)

(A) $1 : 2$ (B) $2 : 3$ (C) $3 : 4$ (D) $8 : 9$

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW booklet. **Marks**

(a) Write $\dfrac{\left(\pi + e^4\right)^{\frac{3}{2}}}{\frac{4}{7}}$ correct to 2 significant figures. **2**

(b) Write the *exact* value of $\tan 330°$. **1**

(c) Find the value of a and b if $\dfrac{1}{5 - \sqrt{12}} \equiv a + b\sqrt{3}$. **2**

(d) Fully factorise: $x^4 - 81$. **2**

(e) Solve $|x - 5| < 3$, and sketch the solution on a number line. **3**

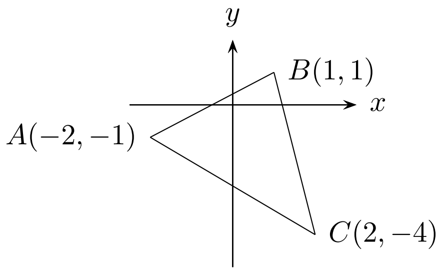
(f) Evaluate $\displaystyle\int (4x + 3)^6 \, dx$, expressing your answer in simplest form. **2**

(g) Find the equation of the normal to the curve $y = 3e^{2x}$ at the point $x = 1$. **3**

Question 12 (15 Marks) Commence a NEW booklet. **Marks**

(a) Evaluate $\sum_{k=2}^{5}(-1)^k\left(\frac{1}{k}\right)$ **2**

(b) The diagram shows the points $A(-2,-1)$, $B(1,1)$ and $C(2,-4)$.



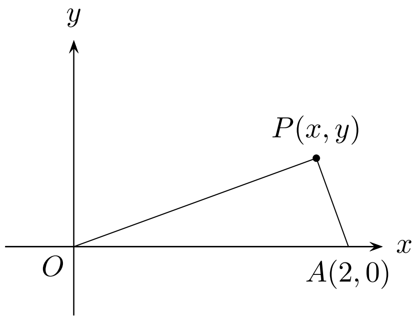
i. Calculate the length of the interval AB. **1**

ii. Find the equation of the line AB. **2**

iii. Show that the perpendicular distance from C to the line AB is $\frac{17}{\sqrt{13}}$. **1**

iv. Hence, calculate the area of $\triangle ABC$. **1**

(c) Refer to the following diagram:



i. Write down the gradient of AP in terms of x and y. **1**

ii. Show that the equation of the locus of all points P, such that $OP \perp AP$ is **2**

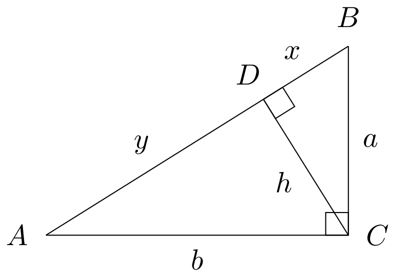
$$x^2 - 2x + y^2 = 0$$

iii. Deduce that the locus of all points P such that $OP \perp AP$ is a circle. Write down the centre and the radius of the circle. **2**

Question 12 continued on the next page...

Question 12 continued from the previous page...

(d) In the diagram $\triangle ABC$ is a right angled at C. The perpendicular from C to AB meets AB at D.



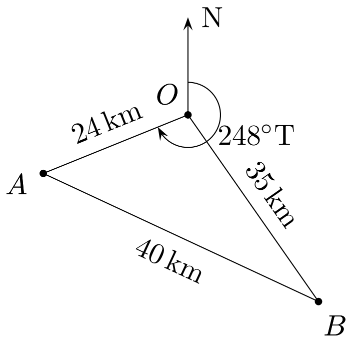
Also, $BC = a$, $AC = b$, $DC = h$, $DB = x$ and $AD = y$.

i. Show that $h(x + y) = ab$. **2**

ii. Show that $h = \dfrac{ab}{\sqrt{a^2 + b^2}}$. **1**

Question 13 (15 Marks) Commence a NEW booklet. **Marks**

(a) A section of rainforest is to be designated for a species count. The shape is shown below. The bearing of landmark A from landmark O is 248°T and is 24 km in distance. The distance from landmark A to B is 40 km and from landmark B to O is 35 km.



i. Show that $\angle AOB = 83°$, to the nearest degree. **2**

ii. Calculate the area of the rainforest, correct to the nearest square kilometre. **2**

iii. What is the bearing of landmark O from landmark B? **1**

(b) Solve for x for $0° \leq x \leq 360°$: $\quad 2\cos^2 x + 3\sin x \cos x + \sin^2 x = 0$ **3**

(c) Express in simplest form:

i. Find: $\dfrac{d}{dx}(\operatorname{cosec} x)$. **2**

ii. Evaluate: $\displaystyle\int_0^{\frac{\pi}{4}} \sin 3x \, dx$. **2**

(d) By differentiating $\sin^3 x + 1$ or otherwise, evaluate: **3**

$$\int \frac{\cos x - \cos^3 x}{\sin^3 x + 1} \, dx$$

Question 14 (15 Marks) Commence a NEW booklet. **Marks**

(a) A parabola has equation $x = -y^2 + 4y - 6$.

i. Find the coordinates of its vertex. **2**

ii. Find the coordinates of its focus and the equation of its directrix. **2**

iii. Sketch the parabola, showing all relevant features. **2**

(b) Carbon emissions since 2010 have been growing exponentially according to the differential equation

$$\frac{dC}{dt} = kC$$

where C is the number of gigatonnes of carbon (GtC) emitted by burning fossil fuels, cement and land use change. (Source: `co2now.org`)

During 2010, 9.19 GtC was emitted throughout the entire world. In 2013, this had risen to 9.9 GtC.

i. Show that $C = C_0 e^{kt}$ is a solution to this differential equation. **1**

ii. Find the value of C_0 and k, correct to 2 decimal places. **3**

iii. Find the number of gigatonnes of carbon emitted in 2030 if no action is taken and assuming that emissions continues to grow at this rate, correct to 2 decimal places. (You may use your 2 decimal place values from previous parts) **1**

iv. Find the rate of increase of carbon emitted during the year 2030. **1**

v. Assuming no reduction in emissions, predict the nearest year when catastrophic climate change occurs if in 2010, there remains a total of 1 000 GtC of carbon that can be emitted into the atmosphere. **3**

Question 15 (15 Marks) Commence a NEW booklet. **Marks**

(a) If α and β are the roots to the equation $2x^2 + 3x + 7 = 0$, evaluate:

i. $\alpha + \beta$. **1**

ii. $(\alpha - 2)(\beta - 2)$. **2**

(b) A function is defined by $f(x) = 2x^3 - 6x + 3$.

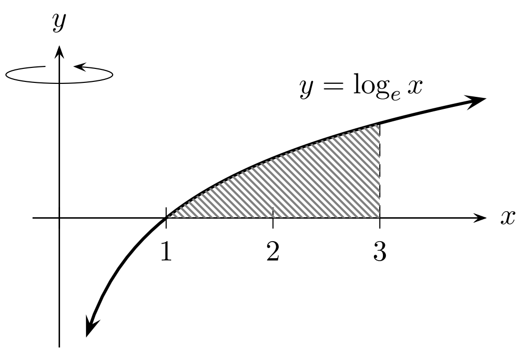
i. Find the coordinates of the turning points of the graph $y = f(x)$ and determine their nature. **3**

ii. Hence sketch the graph of $y = f(x)$, showing the turning points and the y intercept. **2**

(c) Using Simpson's rule with five function values, find an estimate for the definite integral, correct to 2 decimal places: **3**

$$\int_2^6 2^{x-1}\,dx$$

(d) The area bounded by the curve $y = \log_e x$, the x axis and the line $x = 3$ is rotated about the y axis. **4**



Find the exact volume that is generated.

Question 16 (15 Marks) Commence a NEW booklet. **Marks**

(a) A particular bus route saw patronage of 50 passengers on its first day of operation. As the route's popularity grew with passengers, patronage grew steadily by 14 passengers every subsequent day.

i. How many passengers did the bus route carry on its 28th day of operation? **1**

ii. After how many days of operation did the bus route carry over 800 passengers on one single day of operation? **1**

iii. On which day did the 10 000th passenger use the bus route? **2**

(b) Consider the geometric series

$$\sin^2\theta + \sin^2\theta\cos^2\theta + \sin^2\theta\cos^4\theta + \cdots$$

where $0 < \theta < \dfrac{\pi}{2}$.

i. Show that the partial sum S_n of the first n terms is given by **2**

$$S_n = 1 - \cos^{2n}\theta$$

ii. Explain why this geometric series always has a limiting sum. **1**

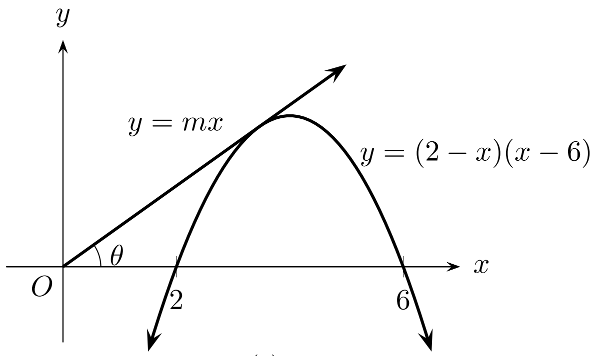
iii. Let S be the limiting sum of this geometric series. Show that **2**

$$S - S_n = \cos^{2n}\theta$$

iv. If $\theta = \dfrac{\pi}{3}$, find the least value of n for which **2**

$$S - S_n < 10^{-6}$$

(c) The line $y = mx$ represents the flight path of a plane which has just taken off from the airport at O. The parabola $y = (2-x)(x-6)$ represents a hill that a plane must fly over.



i. Explain carefully, why the solution(s) to **1**

$$x^2 + (m-8)x + 12 = 0$$

represents the x coordinate where the plane will just clear the hill.

ii. Find the angle of inclination θ which the plane must ascend in order to just clear the hill. **3**

End of paper.

Suggested Solutions

Section I

1. (B) **2.** (D) **3.** (A) **4.** (D) **5.** (C)
6. (A) **7.** (A) **8.** (D) **9.** (A) **10.** (B)

Section II

Question 11 (Lam)

(a) (2 marks)

$$\frac{\left(\pi + e^4\right)^{\frac{3}{2}}}{\frac{4}{7}} = 767.80\cdots = 770 \text{ (2 s.f.)}$$

(b) (1 mark)

$$\tan 330^\circ = -\frac{1}{\sqrt{3}}$$

(c) (2 marks)

$$\begin{aligned}\frac{1}{5-\sqrt{12}} \times \frac{5+\sqrt{12}}{5+\sqrt{12}} &= \frac{5+\sqrt{12}}{25-12}\\ &= \frac{5}{13} + \frac{1}{13}\sqrt{12}\\ &= \frac{5}{13} + \frac{1}{13} \times 2\sqrt{3}\\ &= \frac{5}{13} + \frac{2}{13}\sqrt{3}\end{aligned}$$

$$\therefore a = \frac{5}{13} \qquad b = \frac{2}{13}$$

(d) (2 marks)

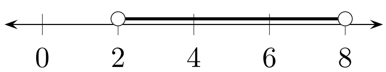
$$\begin{aligned}x^4 - 81 &= \left(x^2-9\right)\left(x^2+9\right)\\ &= (x-9)(x+9)\left(x^2+9\right)\end{aligned}$$

(e) (3 marks)

$$|x-5| < 3$$

$$\underset{+5}{-3} < x - \underset{+5}{5} < \underset{+5}{3}$$

$$2 < x < 8$$



(f) (2 marks)

$$\begin{aligned}\int (4x+3)^6\,dx &= \frac{(4x+3)^7}{7\times 4} + C\\ &= \frac{1}{28}(4x+3)^7 + C\end{aligned}$$

(g) (3 marks)

$$y = 3e^{2x}$$

Finding the gradient of the tangent at $x = 1$:

$$\frac{dy}{dx} = 6e^{2x}\Big|_{x=1} = 6e^2$$

$$\therefore m_\perp = -\frac{1}{6e^2}$$

At $x = 1$, $y = 3e^2$. Apply the point-gradient formula,

$$\frac{y-3e^2}{x-1} = -\frac{1}{6e^2}$$

$$6e^2y - 18e^4 = -x + 1$$

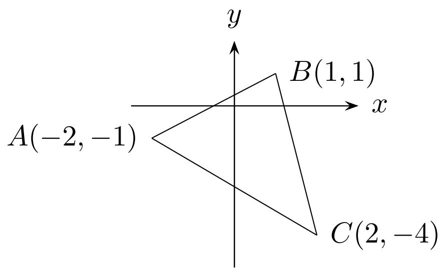
$$\therefore x + 6e^2y - 18e^4 - 1 = 0$$

Question 12 (Tan)

(a) (2 marks)

$$\begin{aligned}\sum_{k=2}^{5}(-1)^k\left(\frac{1}{k}\right) &= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}\\ &= \frac{13}{60}\end{aligned}$$

(b) i. (1 mark)



$$\begin{aligned}AB &= \sqrt{(-2-1)^2 + (-1-1)^2}\\ &= \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}\end{aligned}$$

ii. (2 marks)

$$m_{AB} = \frac{-1-1}{-2-1} = \frac{2}{3}$$

Applying the point gradient formula,

$$\frac{y-1}{x-1} = \frac{2}{3}$$

$$3y - 3 = 2x - 2$$

$$2x - 3y + 1 = 0$$

iii. (1 mark)

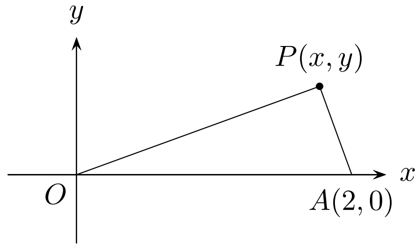
✓ [1] for *showing*, via perpendicular distance formula, the required distance.

$$\begin{aligned} d_\perp &= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} \\ &= \frac{|2(2) - 3(-4) + 1|}{\sqrt{2^2+3^2}} \\ &= \frac{17}{\sqrt{13}} \end{aligned}$$

iv. (1 mark)

$$\begin{aligned} A &= \frac{1}{2} \times AB \times d_\perp \\ &= \frac{1}{2}\sqrt{13} \times \frac{17}{\sqrt{13}} = \frac{17}{2} \end{aligned}$$

(c) i. (1 mark)



$$m_{AP} = \frac{y-0}{x-2} = \frac{y}{x-2}$$

ii. (2 marks)

$$m_{OP} = \frac{y}{x}$$

If $OP \perp AP$, then

$$\begin{aligned} m_{OP} \times m_{AP} &= -1 \\ \frac{y}{x-2} \times \frac{y}{x} &= -1 \\ \frac{y^2}{x(x-2)} &= -1 \\ y^2 &= -x^2 + 2x \\ \therefore x^2 - 2x + y^2 &= 0 \end{aligned}$$

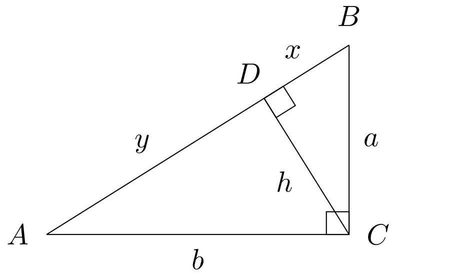
iii. (2 marks)

Completing the square,

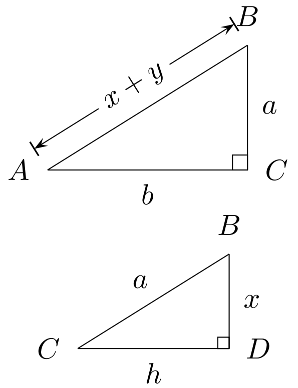
$$\begin{aligned} x^2 - 2x + 1 + y^2 &= 1 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

which represents a circle of centre $(1, 0)$ and radius 1.

(d) i. (2 marks)



Redrawing with $\triangle BCD$ reoriented:



In $\triangle ABC$ and $\triangle BCD$,

- $\angle BCA = \angle BDC = 90^\circ$ (given)
- $\angle ABC = \angle CBD$ (common)

$\therefore \triangle ABC \parallel\!\!| \triangle BCD$ (equiangular). Hence all corresponding sides are in the same ratio, i.e.

$$\begin{aligned} \frac{h}{b} &= \frac{a}{x+y} \\ h(x+y) &= ab \end{aligned}$$

Alternatively, find via area of of $\triangle ABC$:

$$\begin{aligned} A &= \frac{1}{2}h(x+y) \\ A &= \frac{1}{2}ab \\ \therefore \frac{1}{2}h(x+y) &= \frac{1}{2}ab \\ \therefore h(x+y) &= ab \end{aligned}$$

Alternatively, use trigonometric ratios. In $\triangle ABC$,

$$\begin{aligned}\sin\angle ABC &= \frac{b}{(x+y)}\\ \sin\angle CBD &= \frac{h}{a}\\ \sin\angle ABC &= \sin\angle CBD\\ \therefore \frac{b}{x+y} &+ \frac{h}{a}\\ ab &= h(x+y)\end{aligned}$$

ii. (1 mark)

✓ [1] for showing Pythagoras' Theorem usage in $\triangle ABC$ and thus result required.

In $\triangle ABC$, $a^2+b^2=(x+y)^2$, i.e.

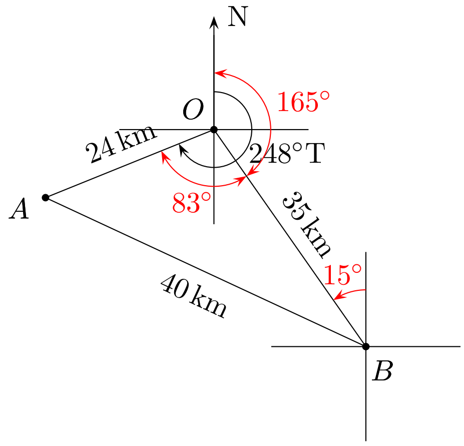
$$x+y=\sqrt{a^2+b^2}$$

Hence,

$$h=\frac{ab}{x+y}=\frac{ab}{\sqrt{a^2+b^2}}$$

Question 13 (Wall)

(a) i. (2 marks)



In $\triangle AOB$, apply the cosine rule:

$$\cos\angle AOB=\frac{24^2+35^2-40^2}{2(24)(35)}=\frac{67}{560}$$

$$\therefore \angle AOB = 83^\circ \text{ (1 d.p.)}$$

ii. (2 marks)

$$\begin{aligned}A &= \frac{1}{2}ab\sin C\\ &\approx \frac{1}{2}\times 24\times 35\times\sin 83^\circ\\ &\approx 417\,\text{km}^2\end{aligned}$$

iii. (1 mark)

$$345^\circ$$

(See red text in diagram)

(b) (3 marks)

$$2\cos^2 x+3\sin x\cos x+\sin^2 x=0$$
$$(2\cos x+\sin x)(\cos x+\sin x)=0$$

$2\cos x+\sin x=0$	$\cos x+\sin x=0$
$\sin x=-2\cos x$	$\sin x=-\cos x$
$\tan x=-2$	$\tan x=-1$

$x=116^\circ 34', 296^\circ 34'$ when $\tan x=-2$

$=135^\circ, 315^\circ$ when $\tan x=-1$

(c) i. (2 marks)

$$\begin{aligned}\frac{d}{dx}(\operatorname{cosec} x) &= \frac{d}{dx}\left((\sin x)^{-1}\right)\\ &= -(\sin x)^{-2}\cos x\\ &\left(=-\frac{\cos x}{\sin^2 x}\right)\\ &(=\cot x\operatorname{cosec} x)\end{aligned}$$

ii. (2 marks)

$$\begin{aligned}\int_0^{\frac{\pi}{4}}\sin 3x\,dx &= -\frac{1}{3}\left[\cos 3x\right]_0^{\frac{\pi}{4}}\\ &= -\frac{1}{3}\left(\cos\frac{3\pi}{4}-\cos 0\right)\\ &= -\frac{1}{3}\left(-\frac{1}{\sqrt{2}}-1\right))\\ &= \frac{1}{3}\left(\frac{1}{\sqrt{2}}+1\right)\end{aligned}$$

(d) (3 marks)

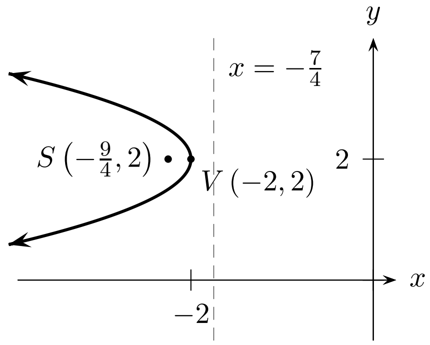
$$\begin{aligned}\frac{d}{dx}\left(\sin^3 x+1\right) &= \frac{d}{dx}\left((\sin x)^3+1\right)\\ &= 3\sin^2 x\cos x\\ &= 3\left(1-\cos^2 x\right)\cos x\\ &= 3\left(\cos x-\cos^3 x\right)\end{aligned}$$

$$\begin{aligned}\therefore \int\frac{\cos x-\cos^3 x}{\sin^3 x+1}\,dx &= \frac{1}{3}\int\frac{3\left(\cos x-\cos^3 x\right)}{\sin^3 x+1}\,dx\\ &= \frac{1}{3}\ln\left(\sin^3 x+1\right)+C\end{aligned}$$

Question 14 (Bhamra)

(a) i. (2 marks)

$$\begin{aligned} x &= -y^2 + 4y - 6 \\ x + 6 &= -\left(y^2 - 4x + 4 - 4\right) \\ x + 6 &= -(y-2)^2 + 4 \\ \therefore x + 2 &= -4\left(\frac{1}{4}\right)(y-2)^2 \end{aligned}$$



Hence vertex is at $(-2, 2)$.

ii. (2 marks)

$$S\left(-\frac{9}{4}, 2\right)$$

$$\text{Directrix } x = -\frac{7}{4}$$

iii. (2 marks)

✓ [1] for shape & direction.

✓ [1] for directrix, S and V.

See above.

(b) i. (1 mark)

$$\begin{aligned} \frac{dC}{dt} &= kC \\ \frac{dC}{C} &= k\,dt \end{aligned}$$

Integrating,

$$\begin{aligned} \int \frac{dC}{C} &= \int k\,dt \\ \log_e C &= kt + C_1 \\ C &= e^{kt + C_1} = e^{kt}e^{C_1} \\ &= C_0 e^{kt} \end{aligned}$$

where $e^{C_1} = C_0$ (constant).

Alternatively differentiate $C = C_0 e^{kt}$ to show required result.

ii. (3 marks)

✓ [1] for value of C_0.

✓ [2] for working and value of k.

Year 2010 corresponds to $t = 0$.

$$\begin{aligned} C &= 9.19 = C_0 e^0 \\ \therefore C_0 &= 9.19 \end{aligned}$$

When $t = 3$, $C = 9.9$:

$$\begin{aligned} 9.9 &= 9.19e^{3k} \\ e^{3k} &= \frac{9.9}{9.19} \\ 3k &= \log_e \frac{9.9}{9.19} \\ \therefore k &= \frac{1}{3}\log_e \frac{9.9}{9.19} \\ &\approx 0.0248\cdots = 0.025 \text{ (3 d.p.)} \end{aligned}$$

iii. (1 mark) When $t = 20$,

$$\begin{aligned} C &= 9.19e^{0.025 \times 20} \\ &= 15.15\cdots \end{aligned}$$

iv. (1 mark)

$$\begin{aligned} \frac{dC}{dt} &= kC \\ &= 0.025 \times 15.15\cdots \\ &= 0.378\,7 \text{ GtC/year} \end{aligned}$$

v. (3 marks)

✓ [1] only for attempts to make a substitution and find the amount emitted during that year.

✓ [2] only for attempting to use some form of arithmetic/geometric series (sum)

✓ [3] (full marks) for integrating and attempts to find t_1.

Total emitted (1 000 GtC) is the integral from $t = 0$ to $t = t_1$, where

$k \approx 0.025$:

$$\begin{aligned} 1\,000 &= \int_0^{t_1} 9.19e^{kt}\,dt \\ &= \frac{9.19}{k}\left[e^{kt}\right]_0^{t_1} \\ &= \frac{9.19}{k}\left(e^{kt_1} - e^0\right) \\ \frac{1\,000k}{9.19} &= e^{kt_1} - 1 \\ e^{kt_1} &= \frac{1\,000k}{9.19} + 1 \\ kt_1 &= \log_e\left(\frac{1\,000k}{9.19} + 1\right) \\ t_1 &= \frac{1}{k}\log_e\left(\frac{1\,000k}{9.19} + 1\right) \\ &= 52.734\cdots \end{aligned}$$

Catastrophic climate change would occur approx 53 years after 2010, i.e. in 2063.

Question 15 (Tan)

(a) i. (1 mark)

$$\begin{aligned} 2x^2 + 3x + 7 &= 0 \\ \alpha + \beta &= -\frac{b}{a} = -\frac{3}{2} \end{aligned}$$

ii. (2 marks)

$$\begin{aligned} \alpha\beta &= \frac{c}{a} = \frac{7}{2} \\ \therefore (\alpha - 2)(\beta - 2) &= \alpha\beta - 2(\alpha + \beta) + 4 \\ &= \frac{7}{2} - 2\left(-\frac{3}{2}\right) + 4 \\ &= \frac{21}{2} \end{aligned}$$

(b) i. (3 marks)

$$\begin{aligned} f(x) &= 2x^3 - 6x + 3 \\ f'(x) &= 6x^2 - 6 \end{aligned}$$

Stationary pts occur when $f'(x) = 0$:

$$\begin{aligned} 6(x^2 - 1) &= 0 \\ (x - 1)(x + 1) &= 0 \\ \therefore x &= \pm 1 \\ f''(x) &= 12x \end{aligned}$$

- When $x = -1$:

$$\begin{aligned} f(-1) &= -2 + 6 + 3 = 7 \\ f''(-1) &= -12 < 0 \end{aligned}$$

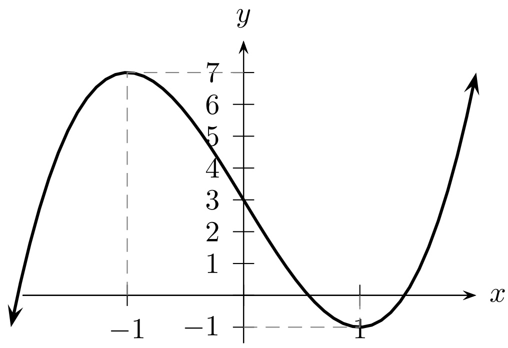
Hence $(-1, 7)$ is a local maximum.

- When $x = 1$:

$$\begin{aligned} f(1) &= 2 - 6 + 3 = -1 \\ f''(1) &= 12 > 0 \end{aligned}$$

Hence $(1, 1)$ is a local minimum.

ii. (2 marks)



(c) (3 marks)

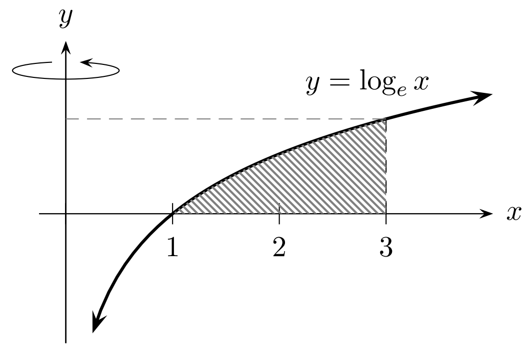
- ✓ [1] for table
- ✓ [1] for substitution into Simpson's Rule.
- ✓ [1] for final answer.

x	2	3	4	5	6
2^{x-1}	2	4	8	16	32

$$\begin{aligned} \int_2^6 2^{x-1}\,dx &\approx \frac{1}{3}\left(2 + 32 + 4(4 + 16) + 2(8)\right) \\ &= \frac{130}{3} \end{aligned}$$

(d) (4 marks)

- ✓ [1] for volume of cylinder
- ✓ [1] for changing subject from y to x
- ✓ [1] for finding volume generated by rotation about y axis.
- ✓ [1] for subtracting volumes.



Volume generated =
Volume of cylinder−Volume around y axis

- Volume of cylinder with radius 3, height $h = \ln 3$:

$$V = \pi r^2 h = 9\pi \ln 3$$

- Volume generated about y axis:

$$\begin{aligned} y &= \log_e x \\ x &= e^y \\ V &= \pi \int_0^{\ln 3} x^2\, dy \\ &= \pi \int_0^{\ln 3} e^{2y}\, dy \\ &= \frac{\pi}{2}\left[e^{2y}\right]_0^{\ln 3} \\ &= \frac{\pi}{2}\left(e^{2\ln 3} - e^0\right) \\ &= \frac{\pi}{2}(9-1) = 4\pi \end{aligned}$$

- Volume generated when shaded area is rotated:

$$V = 9\pi \ln 3 - 4\pi$$

Question 16 (Lam)

(a) i. (1 mark)

$$\begin{aligned} &50, 50+14, 50+14(2)\cdots \\ &a = 50 \qquad d = 14 \\ T_n &= a + d(n-1) \\ &= 50 + 14(n-1) \\ &= 50 + 14n - 14 \\ &= 36 + 14n \\ \therefore T_{28} &= 36 + 14(28) \\ &= 428 \end{aligned}$$

ii. (1 mark)

$$\begin{aligned} T_n &> 800 \\ \therefore 36 + 14n &> 800 \\ 14n &> 764 \\ n &> 54.57\cdots \end{aligned}$$

Carries over 800 passengers on one day on the 55th day.

iii. (2 marks)

$$\begin{aligned} S_n &= 10\,000 \\ 10\,000 &= \frac{n}{2}\left(2a + d(n-1)\right) \\ &= \frac{n}{2}(100 + 14n - 14) \\ &= \frac{n}{2}(86 + 14n) \\ \therefore 7n^2 + 43n &= 10\,000 \\ 7n^2 + 43n - 10\,000 &= 0 \\ n &= \frac{-43 \pm \sqrt{43^2 - 4(7)(-10\,000)}}{2\times 7} \\ &= \frac{-43 \pm \sqrt{281\,849}}{14} \end{aligned}$$

As $n > 0$,

$$n = \frac{-43 + \sqrt{281849}}{14} \approx 34.85$$

The route carried its 10 000th passenger on the 35th day.

(b) i. (2 marks)

$$\sin^2\theta + \sin^2\theta\cos^2\theta + \sin^2\theta\cos^4\theta + \cdots$$

$$a = \sin^2\theta \qquad r = \cos^2\theta$$

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ &= \frac{\sin^2\theta\left(1 - \left(\cos^2\theta\right)^n\right)}{1-\cos^2\theta} \\ &= 1 - \cos^{2n}\theta \end{aligned}$$

ii. (1 mark)

- As the common ratio is $\cos^2\theta$, and $-1 \le \cos\theta \le 1$, then

$$0 \le \cos^2\theta \le 1$$

Since the magnitude of the common ratio is less than 1, a limiting sum exists.

iii. (2 marks)

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{\sin^2\theta}{1-\cos^2\theta} \\ &= 1 \end{aligned}$$

$$\therefore S - S_n = 1 - \left(1 - \cos^{2n}\theta\right) = \cos^{2n}\theta$$

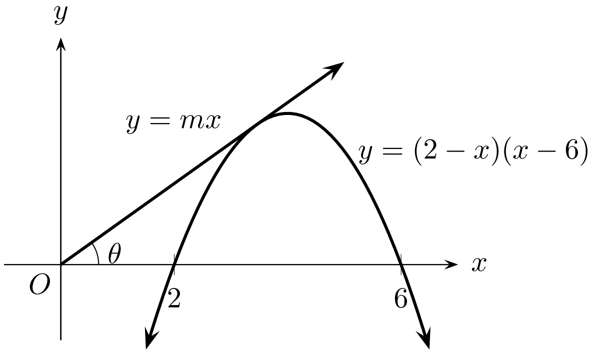
iv. (2 marks) When $\theta = \frac{\pi}{3}$

$$(\cos\theta)^{2n} = \left(\cos\frac{\pi}{3}\right)^{2n} = \left(\frac{1}{2}\right)^{2n}$$

$$\begin{aligned} \therefore \left(\frac{1}{2}\right)^{2n} &< 10^{-6} \\ 2n\log\frac{1}{2} &< \log 10^{-6} \\ 2n &> \frac{\log 10^{-6}}{\log\frac{1}{2}} \\ n &> \frac{\log 10^{-6}}{2\log\frac{1}{2}} \approx 9.96 \end{aligned}$$

$\therefore n = 10$ for this to first occur.

(c) i. (1 mark)



$$\begin{cases} y = mx \\ y = -x^2 + 8x - 12 \end{cases}$$

Solving simultaneously to find the point of intersection,

$$\begin{aligned} mx &= -x^2 + 8x - 12 \\ x^2 + x(m-8) + 12 &= 0 \end{aligned}$$

ii. (3 marks)

✓ [1] for identifying $\Delta = 0$ for the plane to 'just clear' the mountain.

✓ [1] for both values of $\tan\theta$

✓ [1] for final answer.

$\Delta = 0$ in the quadratic in x, involving m:

$$\begin{aligned} \Delta = (m-8)^2 - 4(1)(12) &= 0 \\ (m-8)^2 &= 48 \\ m - 8 &= \pm 4\sqrt{3} \\ \therefore m &= 8 \pm 4\sqrt{3} \end{aligned}$$

As $m = \tan\theta$,

$$\begin{aligned} \therefore \tan\theta &= 8 \pm 4\sqrt{3} \\ \tan\theta &= 1.071 \text{ or } 14.928 \\ \theta &= 46.98^\circ \text{ or } 86.16^\circ \end{aligned}$$

Checking the first derivative at $x = 8 + 4\sqrt{3}$:

$$\begin{aligned} y &= -x^2 + 8x - 12 \\ \frac{dy}{dx} &= -2x + 8 \end{aligned}$$

If $m = 8 + 4\sqrt{3}$,

$$\begin{aligned} 8 + 4\sqrt{3} &= -2x + 8 \\ \therefore -2x &= 4\sqrt{3} \end{aligned}$$

$x = -2\sqrt{3}$ (which is not possible as $x > 0$)

If $m = 8 - 4\sqrt{3}$,

$$\begin{aligned} 8 - 4\sqrt{3} &= -2x + 8 \\ \therefore 2x &= 4\sqrt{3} \\ x &= 2\sqrt{3} \end{aligned}$$

Hence angle of inclination θ is 46.98°.